Lecture 31

11.11 - Applications of Taylor Series

In Calc I, you did linear approximations of a function near a:

$$f(x) \approx f(a) + f'(a)(x-a) \quad \text{for } x \text{ close to } a$$

This is approximating $f(x)$ by its first Taylor polynomial $T_1(x)$. This is sometimes called a first order approximation of f. As might be expected, if we take more terms of the Taylor series, we get a better approximation:

$$f(x) \approx f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

near a. This is an n^{th} order approximation. Also, taking larger n usually allows us to expand the range of x values on which the approximation has an acceptable error.

See Mathematica code for approximation of e^x

When we approximate $f(x)$ by $T_n(x)$, the error is $|R_n(x)| = |f(x) - T_n(x)|$, which we have two possible ways to compute:

1) (Always Works) Taylor's Inequality

If $|f^{(n+1)}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M|x-a|^{n+1}}{(n+1)!} \quad \text{for } |x-a| \leq d.$$

2) If the Taylor series is an alternating series, we can use the error estimates from that:

$$S = \sum_{k=1}^{\infty} (-1)^{k-1} b_k \quad , \quad S_n = \sum_{k=1}^{n} (-1)^{k-1} b_k$$

$$|R_n| = |S - S_n| \leq b_{n+1}$$

Ex: Consider $f(x)=e^x$ and its Maclauren series.

ⓐ What is the 4th order Taylor approximation of e^x? $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $(a=0)$

$$T_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$$

ⓑ How accurate is this approximation on $[-4,4]$?

Taylor's Ineq.: $|R_4(x)| \le \frac{M|x|^5}{5!} = \frac{e^4|x|^5}{5!} \le \frac{e^4 4^5}{5!} \approx 466$

$|f^{(5)}(x)| = |e^x| \le e^4 = M$ on $[-4,4]$

ⓒ Find an interval around 0 (e.g. $[-d,d]$) on which the error be made less than 10^{-3}.

$|R_4(x)| \le \frac{e^d d^5}{5!} = 10^{-3} \rightsquigarrow d \le 0.58$ will work

(solved for with computer)

$|f^{(5)}(x)| = |e^x| \le e^d$ on $[-d,d]$

Ex: Consider again $f(x)=e^x$.

(a) What is the 3^{rd} Taylor polynomial of $f(x)$ at $a=2$?

$$T_3(x) = f(2) + f'(2)(x-2) + \frac{f''(2)}{2}(x-2)^2 + \frac{f'''(2)}{6}(x-2)^3$$

$$= e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3$$

(b) What is an upper bound on the error of approximating e^x with this on $[1,3]$?

$$|R_3(x)| \le \frac{M|x-2|^4}{4!} = \frac{e^3|x-2|^4}{4!} \le \frac{e^3}{4!} \approx 0.837$$

$$|f^{(4)}(x)| = |e^x| \le e^3 \text{ on } [1,3]$$

since $|x-2|^4 \le 1$ on $[1,3]$

Ex: ⓐ Approximate $f(x) = \cos x$ to 3^{rd} order at $a = \frac{\pi}{2}$.

$$T_3(x) = f(\tfrac{\pi}{2}) + f'(\tfrac{\pi}{2})(x-\tfrac{\pi}{2}) + \frac{f''(\frac{\pi}{2})}{2}(x-\tfrac{\pi}{2})^2 + \frac{f'''(\frac{\pi}{2})}{6}(x-\tfrac{\pi}{2})^3$$

$$= 0 - (x-\tfrac{\pi}{2}) + 0 + \frac{1}{6}(x-\tfrac{\pi}{2})^3$$

$$= -(x-\tfrac{\pi}{2}) + \frac{1}{6}(x-\tfrac{\pi}{2})^3$$

(Notice, if we continue this and find the whole series, we get:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(x-\frac{\pi}{2})^{2n+1}}{(2n+1)!}$$
)

ⓑ The Taylor series for $\cos x$ is alternating. Use the alternating series error estimation to find the maximum error on $[\frac{\pi}{4}, \frac{3\pi}{4}]$.

The next term in the series after $\frac{1}{6}(x-\frac{\pi}{2})^3$ is $\frac{-1}{120}(x-\frac{\pi}{2})^5$.

So, using the alternating series estimation theorem:

$$|R_3(x)| \le \frac{1}{120}\left|x-\frac{\pi}{2}\right|^5 \le \frac{1}{120}\left(\frac{\pi}{4}\right)^5 \text{ on } \left[\frac{\pi}{4}, \frac{3\pi}{4}\right].$$